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Accuracy of semiGLS stabilization of FEM for solving Navier–Stokes equations and *a posteriori* error estimates

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SUMMARY

Stabilization of the finite element method for flow problems at high Reynolds numbers is the main subject of presented research. The semiGLS method is recalled as a modification of the Galerkin least-squares method. The presented work extends our previous paper on this method by its other important aspects. The main aim of this paper is to analyse and comment on the accuracy of the method. *A posteriori* error estimates for incompressible Navier–Stokes equations are used as the main tool for error analysis and some conclusions concerning accuracy are derived. Several numerical experiments are presented for both benchmark and practical problems. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The goal of our study is to modify the established finite element method (FEM) for flows at higher Reynolds numbers. We deal with a numerical solution of incompressible viscous flow. Owing to problems with stability of the numerical method for solving flows at high Reynolds numbers, a number of researchers are interested in improvement of the method. Stabilization techniques for the FEM are commonly accepted tools for achieving better stability nowadays.

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In [1], the *semiGLS* method was introduced as a modification of the Galerkin least-squares (GLS) method studied earlier by Hughes, Franca and their co-workers (e.g. [2–4], refer to [1] for other references).

The presented paper focuses mainly on the aspect of accuracy of the *semiGLS* method. As has been observed earlier, we pay for the better stability of the method by some loss of accuracy.

A straightforward way for evaluation of this loss was described in [1]. It is based on comparison of discrete norms of approximate solutions obtained with and without stabilization.

The approach presented in this paper uses *a posteriori* error estimates for evaluation of the error caused by the stabilization technique. We aim at finding the distribution of error for large Reynolds numbers. For low-speed flows, we estimate the error and compare its distribution for both basic and stabilized methods.

2. MODEL PROBLEM

Let Ω be an open-bounded domain in $\mathbb{R}^2$ filled with an incompressible viscous fluid, and let Γ be its boundary. Isothermal steady (time-independent) flow of such fluid is governed by the following Navier–Stokes system of partial differential equations (nonconservative form)

$$(\mathbf{u}\cdot\nabla)\mathbf{u}-\nu\Delta\mathbf{u}+\nabla p=\mathbf{f} \quad \text{in } \Omega \tag{1}$$

$$\nabla\cdot\mathbf{u}=0 \quad \text{in } \Omega \tag{2}$$

$$\mathbf{u}=\mathbf{g} \quad \text{on } \Gamma_g \tag{3}$$

$$-\nu(\nabla\mathbf{u})\mathbf{n}+p\mathbf{n}=\mathbf{0} \quad \text{on } \Gamma_h \tag{4}$$

where $\mathbf{u}=(u_1,u_2)^{\mathrm{T}}$ denotes the vector of flow velocity; p denotes the pressure normalized by the density; ν denotes the kinematic viscosity of the fluid (supposed constant); $\mathbf{f}$ denotes the density of volume forces per mass unit; Γ_g and Γ_h are two subsets of Γ satisfying $\overline{\Gamma}=\overline{\Gamma}_g\cup\overline{\Gamma}_h$; $\mathbf{n}$ denotes an outer normal vector to the boundary Γ with unit length; $\mathbf{g}$ is a given function satisfying $\int_\Gamma \mathbf{g}\cdot\mathbf{n}\,\mathrm{d}\Gamma=0$ in the case of $\Gamma=\Gamma_g$.

3. APPROXIMATION OF THE PROBLEM BY FEM

Using the framework of variational formulation of problem (1)–(4) and of mixed FEM, the following *discrete weak steady Navier–Stokes problem* is derived [5]:

Find $\mathbf{u}_h\in V_{gh}$ and $p_h\in Q_h$, satisfying

$$\int_\Omega(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h\cdot\mathbf{v}_h\,\mathrm{d}\Omega+\nu\int_\Omega\nabla\mathbf{u}_h:\nabla\mathbf{v}_h\,\mathrm{d}\Omega-\int_\Omega p_h\nabla\cdot\mathbf{v}_h\,\mathrm{d}\Omega=\int_\Omega\mathbf{f}\cdot\mathbf{v}_h\,\mathrm{d}\Omega \quad \forall\mathbf{v}_h\in V_h \tag{5}$$

$$\int_\Omega\psi_h\nabla\cdot\mathbf{u}_h\,\mathrm{d}\Omega=0 \quad \forall\psi_h\in Q_h \tag{6}$$

$$\mathbf{u}_h-\mathbf{u}_{gh}\in V_h \tag{7}$$

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where for the application of Taylor–Hood finite elements P_2P_1 and/or Q_2Q_1

$$V_{gh}=\{\mathbf{v}_h=(v_{h_1},v_{h_2})^{\mathrm{T}}\in[\mathscr{C}(\overline{\Omega})]^2;\ v_{h_i}|_{T_K}\in R_2(\overline{T_K}),\ K=1,\dots,N,\ i=1,2,\ \mathbf{v}_h=\mathbf{g}\ \text{in nodes on } \Gamma_g\}$$

$$Q_h=\{\psi_h\in\mathscr{C}(\overline{\Omega});\ \psi_h|_{T_K}\in R_1(\overline{T_K}),\ K=1,\dots,N\}$$

$$V_h=\{\mathbf{v}_h=(v_{h_1},v_{h_2})^{\mathrm{T}}\in[\mathscr{C}(\overline{\Omega})]^2;\ v_{h_i}|_{T_K}\in R_2(\overline{T_K}),\ K=1,\dots,N,\ i=1,2,\ \mathbf{v}_h=\mathbf{0}\ \text{in nodes on } \Gamma_g\}$$

Here

$$R_m(\overline{T_K})=\begin{cases}P_m(\overline{T_K}) & \text{if } T_K \text{ is a triangle}\\ Q_m(\overline{T_K}) & \text{if } T_K \text{ is a quadrilateral}\end{cases}$$

is an abbreviated notation for polynomial spaces of degree m on individual elements and $\mathscr{C}(\overline{\Omega})$ denotes the space of continuous functions on $\overline{\Omega}$.

4. semiGLS STABILIZED FORMULATION

We recall the *semiGLS* stabilization technique, which was derived in [1] as a modification of GLS method, proposed by Hughes *et al*. [4]. Applying this stabilization to the momentum equation (5) and adding the continuity equation (6), we introduce the stabilized problem:

Find $\mathbf{u}_h\in V_{gh}$ and $p_h\in Q_h$ satisfying

$$B_{\mathrm{sGLS}}(\mathbf{u}_h,p_h;\mathbf{v}_h,\psi_h)=L_{\mathrm{sGLS}}(\mathbf{v}_h,\psi_h)\quad\forall\mathbf{v}_h\in V_h,\ \ \forall\psi_h\in Q_h \tag{8}$$

$$\mathbf{u}_h-\mathbf{u}_{gh}\in V_h \tag{9}$$

where

$$\begin{aligned}B_{\mathrm{sGLS}}(\mathbf{u}_h,p_h;\mathbf{v}_h,\psi_h)\equiv&\int_\Omega(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h\cdot\mathbf{v}_h\,\mathrm{d}\Omega\\&+v\int_\Omega\nabla\mathbf{u}_h:\nabla\mathbf{v}_h\,\mathrm{d}\Omega-\int_\Omega p_h\nabla\cdot\mathbf{v}_h\mathrm{d}\Omega+\int_\Omega\psi_h\nabla\cdot\mathbf{u}_h\,\mathrm{d}\Omega\\&+\sum_{K=1}^N\int_{T_K}[(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h-v\Delta\mathbf{u}_h+\nabla p_h]\cdot\tau[(\mathbf{u}_h\cdot\nabla)\mathbf{v}_h-v\Delta\mathbf{v}_h+\nabla\psi_h]\,\mathrm{d}\Omega\end{aligned}$$

$$L_{\mathrm{sGLS}}(\mathbf{v}_h,\psi_h)\equiv\int_\Omega\mathbf{f}\cdot\mathbf{v}_h\mathrm{d}\Omega+\sum_{K=1}^N\int_{T_K}\mathbf{f}\cdot\tau[(\mathbf{u}_h\cdot\nabla)\mathbf{v}_h-v\Delta\mathbf{v}_h+\nabla\psi_h]\,\mathrm{d}\Omega$$

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Here τ is a positive stabilization parameter. Refer to [1] for a detailed description of its computation, which is based on ideas from [3].

Although stabilization terms should vanish in the limit for $h \to 0$ (the exact solution) satisfying the formal consistency, the approximate solution does not reach this limit. Hence, these terms remain present in the practically solved equations and modify them slightly. We hypothesize that this is the source of the loss of accuracy.

5. *A POSTERIORI* ERROR ESTIMATES

For evaluating the achieved accuracy of our solution, we use the following error estimator that represents the relative error on element T_K:

$$\mathscr{R}^2(u_{1h}, u_{2h}, p_h, T_K) = \frac{|\Omega| \ \mathscr{E}^2(u_{1h}, u_{2h}, p_h, T_K)}{|T_K| \, \|(u_{1h}, u_{2h}, p_h)\|^2_{V,\Omega}} \tag{10}$$

based on *a posteriori* error estimates in the following form derived for Taylor–Hood elements in [6]

$$\|(e_{u_1}, e_{u_2}, e_p)\|^2_{V,T_K} \leqslant \mathscr{E}^2(u_{1h}, u_{2h}, p_h, T_K) \tag{11}$$

where (u_1, u_2, p) denotes an exact solution; (u_{1h}, u_{2h}, p_h) denotes an approximate solution computed by FEM; $(e_{u_1}, e_{u_2}, e_p) = (u_1 - u_{1h}, u_2 - u_{2h}, p - p_h)$ denotes an error of approximate solution; $\|u_{1h}, u_{2h}, p_h\|^2_{V,\Omega} = \|u_{1h}, u_{2h}\|^2_{1,\Omega} + \|p_h\|^2_{0,\Omega}$, where $\|u_{1h}, u_{2h}\|_{1,\Omega}$ means the Sobolev $H^1(\Omega)$ norm, $\|p_h\|_{0,\Omega}$ means the $L_2(\Omega)$ norm, $|\Omega|, |T_K|$ mean the area of the domain Ω and the element T_K, respectively.

The term on the right-hand side of inequality (11) is evaluated as

$$\begin{aligned}&\mathscr{E}^2(u_{1h}, u_{2h}, p_h, T_K)\\ &\quad = C\left[h_K^2 \int_{T_K} (r_1^2(u_{1h}, u_{2h}, p_h) + r_2^2(u_{1h}, u_{2h}, p_h))\,\mathrm{d}\Omega + \int_{T_K} r_3^2(u_{1h}, u_{2h}, p_h)\,\mathrm{d}\Omega\right]\end{aligned}$$

where

$$r_1(u_{1h}, u_{2h}, p_h) = f_{x_1} - \left(u_{1h}\frac{\partial u_{1h}}{\partial x_1} + u_{2h}\frac{\partial u_{1h}}{\partial x_2}\right) + \nu\left(\frac{\partial^2 u_{1h}}{\partial x_1^2} + \frac{\partial^2 u_{1h}}{\partial x_2^2}\right) - \frac{\partial p_h}{\partial x_1}$$

$$r_2(u_{1h}, u_{2h}, p_h) = f_{x_2} - \left(u_{1h}\frac{\partial u_{2h}}{\partial x_1} + u_{2h}\frac{\partial u_{2h}}{\partial x_2}\right) + \nu\left(\frac{\partial^2 u_{2h}}{\partial x_1^2} + \frac{\partial^2 u_{2h}}{\partial x_2^2}\right) - \frac{\partial p_h}{\partial x_2}$$

$$r_3(u_{1h}, u_{2h}, p_h) = \frac{\partial u_{1h}}{\partial x_1} + \frac{\partial u_{2h}}{\partial x_2}$$

stand for residuals of the system (1)–(2), $h_K = \mathrm{diam}(T_K)$. The constant C is a delicate task in *a posteriori* error estimates. We refer to [6], where we show its derivation for the case of non-stabilized FEM. In this paper we use the constant in a relative sense: we apply the *a posteriori* error estimates to show the relative error on finite elements, in order to show the distribution of the

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error in the solution domain. For the purpose of comparison of this distribution obtained without stabilization and by *semiGLS*, it is important to use the same constant for both solutions.

6. NUMERICAL RESULTS

Comparison of *a posteriori* error estimates for the problem of the lid-driven cavity (see [1] for details) at Reynolds number 10 000 is presented in Figure 1. In Figures 2–3, we can observe results for a channel with an abrupt extension of diameter.

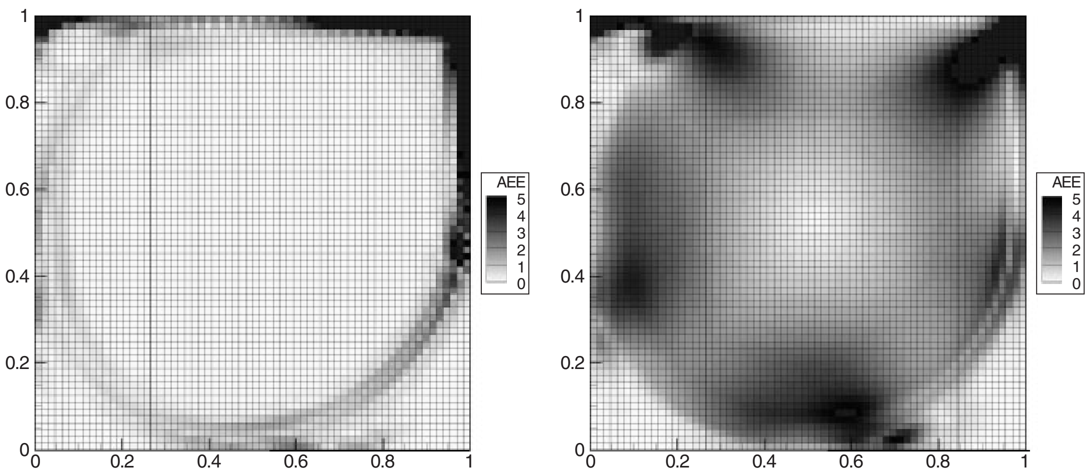

Figure 1. *A posteriori* errors on elements, cavity problem, $Re=10\,000$, uniform mesh 64×64 without stabilization (left) and by *semiGLS* method (right).

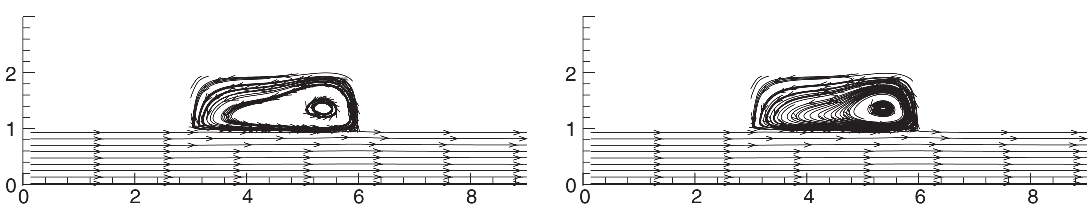

Figure 2. Streamlines in a channel by the method without stabilization (left) and by *semiGLS* method (right), $Re=1000$.

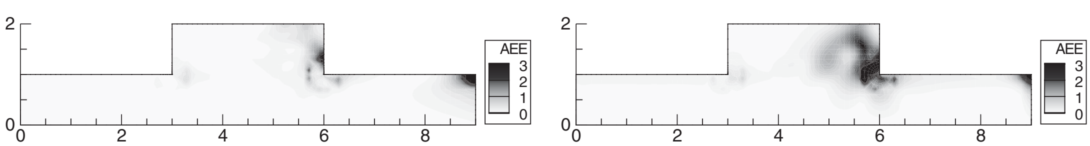

Figure 3. *A posteriori* error estimates in the channel for the Newton method without stabilization (left) and by the *semiGLS* method (right), $Re=1000$.

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Flow past NACA 0012 airfoil (Figures 4 and 5) is investigated as a more practical application. The computational mesh for this problem was described in [1] with results of *unsteady* flow for Reynolds numbers 1000 and 100 000. Some other results were presented in [7]. In this paper, solution of the *steady* flow for Reynolds number 100 is presented and the error is analysed by *a posteriori* error estimates.

In presented plots, AEE is an abbreviation for *a posteriori error estimator* from Equation (10). Let us note that, as was already suggested at the end of Section 5, our error estimator does not yield the exact value of the approximation error. Still it is satisfactory enough to detect elements where the inaccuracy is higher compared with other elements, cf. Figures 1, 3 and 5. Also, though the examples presented (Figures 1–3) are quite hard to compute numerically (singularities at corners, e.g.), analysis of the results shows where to refine the mesh to improve accuracy.

To present such comparisons, Reynolds number for all these experiments was restricted to values, for which we are able to obtain solution also by Galerkin method without stabilization.

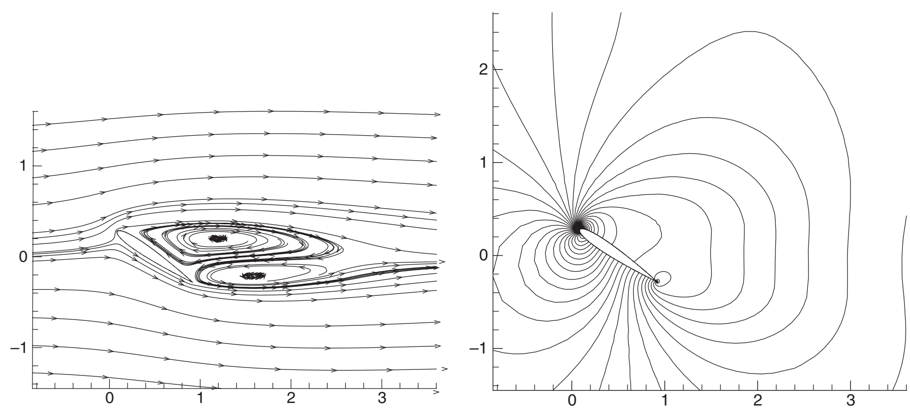

Figure 4. Streamlines (left) and pressure contours (right), $Re = 100$.

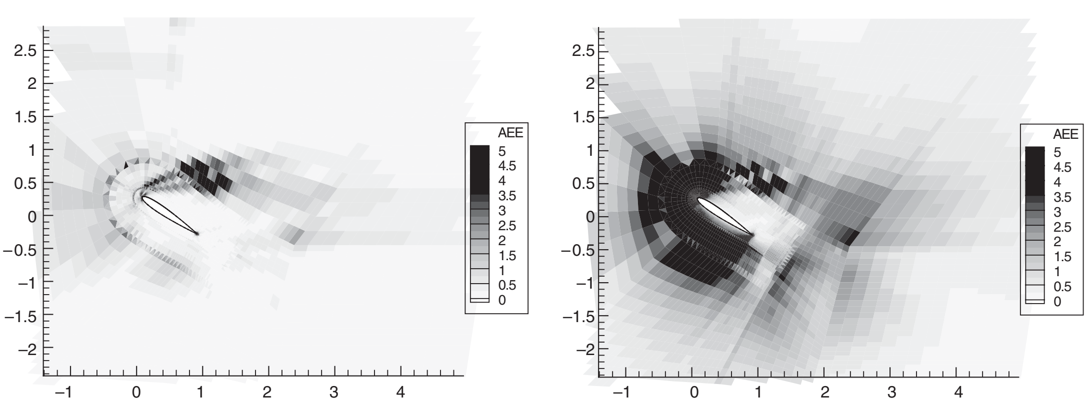

Figure 5. *A posteriori* error on elements for the Newton method without stabilization (left) and by the *semiGLS* method (right), $Re = 100$.

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It does not depreciate an important advantage of this evaluation—we are able to obtain an idea about the error distribution for any Reynolds number, for which the stabilized method converges.

7. CONCLUSION

We have investigated the aspect of accuracy of the *semiGLS* stabilization technique for incompressible Navier–Stokes equations, a modification of the GLS method introduced in [1]. The method has been applied not only to benchmark problems but also to practical problems of external aerodynamics, including problems at higher Reynolds numbers.

The loss of accuracy is inherited in the stabilized method and could be hardly suppressed. It was shown in [1] that it can range from negligible effects to serious influence including qualitative changes of flow such as forming vortex structures. It was also shown in the reference that these effects are reduced with mesh refinement.

Using the new approach presented in this paper, we can evaluate the distribution of the error and therefore obtain the idea about its effect. Moreover, we are able to refine the mesh adaptively in the problematic areas and recompute the solution in the way that is commonly used in connection with *a posteriori* error estimates if a uniform error distribution is desired. Such an approach was presented in [6] for the FEM without stabilization.

In Figures 1, 3 and 5, we can observe that the error can spread to a larger area when using *semiGLS* than for the standard FEM. Similar behaviour was observed also for other stabilization techniques, such as stream line-upwind/Petrov–Galerkin method.

Although the *a posteriori* error estimates offer an important new insight into the distribution of the error and its behaviour, still the problem of accuracy in the stabilized version of the FEM needs deeper understanding and deserves further analysis.

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